

A COMPACTNESS RESULT IN APPROACH THEORY WITH AN APPLICATION TO THE CONTINUITY APPROACH STRUCTURE

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Dedicated to Eva Colebunders on the occasion of her 65th birthday

ABSTRACT. We establish a compactness result in approach theory which we apply to obtain a generalization of Prokhorov's Theorem for the continuity approach structure.

1. INTRODUCTION

Measures of non-compactness ([BG80]) have been studied extensively in the context of approach theory ([L15]), both on an abstract level ([BL94], [BL95]) as in specific approach settings in e.g. hyperspace theory ([LS00']), functional analysis ([LS00]), function spaces ([L04]) and probability theory ([BLV11]). The presence of a vast literature on the interplay between compactness and approach theory is explained by the fact that the latter is a canonical setting which allows for a unified treatment of the classical concept of measure of non-compactness ([L88]).

In this paper we contribute to the knowledge on the interplay between compactness and approach theory. In Section 2 we provide a new compactness result for a general approach space. In Section 3 we apply this result to the specific setting of the so-called continuity approach structure ([BLV13], [L15]) to obtain a quantitative generalization of Prokhorov's Theorem.

2. A COMPACTNESS RESULT IN APPROACH THEORY

Let X be an approach space with approach system $\mathcal{A} = (\mathcal{A}_x)_{x \in X}$. We first recall some notions related to compactness in X . For more details the reader is referred to [L15].

We say that X is *locally countably generated* iff there exists a basis $(\mathcal{B}_x)_{x \in X}$ for \mathcal{A} such that each \mathcal{B}_x is countable.

For $x \in X$, $\phi \in \mathcal{A}_x$ and $\epsilon > 0$ we define the ϕ -ball with center x and radius ϵ as the set $B_\phi(x, \epsilon) = \{y \in X \mid \phi(y) < \epsilon\}$. More loosely, we also refer to the latter set as a ball with center x or a ball with radius ϵ .

Consider a point $x \in X$, a sequence $(x_n)_n$ in X and $\epsilon > 0$. We say that $(x_n)_n$ is ϵ -convergent to x iff each ball B with center x and radius ϵ contains

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x_n for all n larger than a certain n_B . We write $x_n \xrightarrow{\epsilon} x$ to indicate that $(x_n)_n$ is ϵ -convergent to x . We define the *limit operator* of $(x_n)_n$ at x as

$$\lambda(x_n \rightarrow x) = \inf \left\{ \alpha > 0 \mid x_n \xrightarrow{\alpha} x \right\}.$$

We call X *sequentially complete* iff it holds for each sequence $(x_n)_n$ in X that $\inf_{x \in X} \lambda_A(x_n \rightarrow x) = 0$ implies the existence of a point x_0 to which $(x_n)_n$ converges (in the topological coreflection).

Let $A \subset X$ be a set. We say that A is ϵ -*relatively sequentially compact* iff every sequence in A contains a subsequence which is ϵ -convergent and we define the *relative sequential compactness index* of A as

$$\chi_{rsc}(A) = \inf \{ \alpha > 0 \mid A \text{ is } \alpha\text{-relatively sequentially compact} \}.$$

Notice that relatively sequentially compact sets (in the topological coreflection) have relative sequential compactness index zero, but that the converse does not necessarily hold.

If $(\Phi = (\phi_x)_x) \in \Pi_{x \in X} \mathcal{A}_x$, then a set $B \subset X$ is called a Φ -*ball* iff there exist $x \in X$ and $\alpha > 0$ such that $B = B_{\phi_x}(x, \alpha)$. We call A ϵ -*relatively compact* iff it holds for each $\Phi \in \Pi_{x \in X} \mathcal{A}_x$ that A can be covered with finitely many Φ -balls with radius ϵ and we define the *relative compactness index* of A as

$$\chi_{rc}(A) = \inf \{ \alpha > 0 \mid A \text{ is } \alpha\text{-relatively compact} \}.$$

We say that X is ϵ -*Lindelöf* iff it holds for each $\Phi \in \Pi_{x \in X} \mathcal{A}_x$ that X can be covered with countably many Φ -balls with radius ϵ and we define the *Lindelöf index* of X as

$$\chi_L(X) = \inf \{ \alpha > 0 \mid X \text{ is } \alpha\text{-Lindelöf} \}.$$

Theorem 2.2, the main result of this section, interconnects the above notions. For its proof we use the following well-known lemma which belongs to the heart of approach theory ([L15]).

Lemma 2.1 (Lowen). *Let \mathcal{D}_A be the set of quasi-metrics d on X with the property that $d(x, \cdot) \in \mathcal{A}_x$ for each $x \in X$. Then the assignment of collections*

$$\mathcal{B}_{\mathcal{D}_A, x} = \{d(x, \cdot) \mid d \in \mathcal{D}_A\}$$

is a basis for \mathcal{A} .

Theorem 2.2. *Let X be locally countably generated. Then, for any set $A \subset X$,*

$$\chi_{rsc}(A) \leq \chi_{rc}(A) \leq \chi_{rsc}(A) + \chi_L(X).$$

In particular, if $\chi_L(X) = 0$, then

$$\chi_{rsc}(A) = \chi_{rc}(A).$$

If, in addition, X is sequentially complete, then

$$A \text{ is relatively sequentially compact} \Leftrightarrow \chi_{rsc}(A) = 0.$$

Proof. Suppose that A is not ϵ -relatively sequentially compact and fix $\epsilon_0 < \epsilon$. Then there exists a sequence $(a_n)_n$ in A without ϵ -convergent subsequence. But then, for each $x \in X$, there exists $\phi_x \in \mathcal{A}_x$ such that the ball $B_{\phi_x}(x, \epsilon_0)$ contains at most finitely many terms of $(a_n)_n$. Indeed, if this was not the case, then the fact that X is locally countably generated would allow us to

extract an ϵ -convergent subsequence from $(a_n)_n$. Put $\Phi = (\phi_x)_x$. Now one easily sees that A cannot be covered with finitely many Φ -balls with radius ϵ_0 . We conclude that A is not ϵ_0 -relatively compact. We have shown that

$$\chi_{rsc}(A) \leq \chi_{rc}(A).$$

Furthermore, let X be δ -Lindelöf and let A fail to be ϵ -relatively compact, with $\delta < \epsilon$, and fix $\delta < \epsilon_0 < \epsilon$. Then Lemma 2.1 enables us to choose $\Phi \in \Pi_{x \in X} \mathcal{A}_x$ of the form

$$\Phi = (d_x(x, \cdot))_x,$$

where each d_x is a quasi-metric in \mathcal{D}_A , such that X cannot be covered with finitely many Φ -balls with radius ϵ_0 . However, X being δ -Lindelöf, there is a countable cover $(B_n)_n$ of X with Φ -balls with radius δ , say with centers $(x_n)_n$. Now construct a sequence $(a_n)_n$ in A such that, for each n , the Φ -ball with center x_n and radius ϵ_0 contains at most finitely many terms of $(a_n)_n$. But then $(a_n)_n$ has no $(\epsilon_0 - \delta)$ -convergent subsequence. Indeed, if any subsequence $(a_{k_n})_n$ was $(\epsilon_0 - \delta)$ -convergent to x , then we could choose n_0 such that $x \in B_{n_0}$, and then it is not hard to see that the Φ -ball with center x_{n_0} and radius ϵ_0 would contain infinitely many terms of $(a_n)_n$. We conclude that A is not $(\epsilon_0 - \delta)$ -relatively sequentially compact. We have established that

$$\chi_{rc}(A) \leq \chi_{rsc}(A) + \chi_L(X).$$

Suppose, in addition, that X is sequentially complete. Let $\chi_{rsc}(A) = 0$. Fix a sequence $(a_n)_n$ in A and carry out the following construction: Choose a subsequence $(a_{k_1(n)})_n$ and a point $x_1 \in X$ such that

$$\lambda(a_{k_1(n)} \rightarrow x_1) \leq 1.$$

Choose a further subsequence $(a_{k_1 \circ k_2(n)})_n$ and a point $x_2 \in X$ such that

$$\lambda(a_{k_1 \circ k_2(n)} \rightarrow x_2) \leq 1/2.$$

...

Choose a further subsequence $(a_{k_1 \circ \dots \circ k_m(n)})_n$ and a point $x_m \in X$ such that

$$\lambda(a_{k_1 \circ \dots \circ k_m(n)} \rightarrow x_m) \leq 1/m.$$

...

Then it holds for the diagonal subsequence

$$(a'_n = a_{k_1 \circ \dots \circ k_n(n)})_n$$

that $\inf_{x \in X} \lambda(a'_n \rightarrow x) = 0$. Now the sequential completeness of X allows us to conclude that $(a'_n)_n$ is convergent. We infer that A is relatively sequentially compact. \square

3. COMPACTNESS FOR THE CONTINUITY APPROACH STRUCTURE

We start by recalling some basic concepts. They can be found in any standard work on probability theory (e.g. [K02]).

A *cumulative distribution function* (cdf) is a non-decreasing and right-continuous map $F : \mathbb{R} \rightarrow \mathbb{R}$ for which $\lim_{x \rightarrow -\infty} F(x) = 0$ and $\lim_{x \rightarrow \infty} F(x) = 1$.

The collection of (continuous) cdf's is denoted as $\mathcal{F}_{(c)}$.

The *weak topology* \mathcal{T}_w on \mathcal{F} is the initial topology for the source

$$\left(\mathcal{F} \rightarrow \mathbb{R} : F \mapsto \int_{-\infty}^{\infty} h(x) dF(x) \right)_{h \in \mathcal{C}_b(\mathbb{R}, \mathbb{R})}$$

with $\mathcal{C}_b(\mathbb{R}, \mathbb{R})$ the set of bounded and continuous maps $h : \mathbb{R} \rightarrow \mathbb{R}$.

The *uniform distance* between F and G in \mathcal{F} is

$$D_u(F, G) = \sup_{x \in \mathbb{R}} |F(x) - G(x)|.$$

The *convolution product* of F and G in \mathcal{F} is the cdf

$$F \star G = \int_{-\infty}^{\infty} F(\cdot - y) dG(y).$$

This product is commutative and $F \star G \in \mathcal{F}_c$ if $F \in \mathcal{F}_c$.

The following classical result, which can be found in [B82], shows how the previous concepts are interconnected. We denote the underlying topology of D_u as \mathcal{T}_{D_u} .

Theorem 3.1 (Bergström). *The source*

$$((\mathcal{F}, \mathcal{T}_w) \rightarrow (\mathcal{F}_c, \mathcal{T}_{D_u}) : F \mapsto F \star G)_{G \in \mathcal{F}_c}$$

is initial.

In order to lay the structural foundations of the quantitative central limit theory developed in [BLV13], approach theory was invoked. More precisely, the following definition was proposed in [BLV13] (appendix B). It is clearly inspired by Theorem 3.1. We denote the underlying approach structure of D_u as \mathcal{A}_{D_u} .

Definition 3.2. *The continuity approach structure \mathcal{A}_c on \mathcal{F} is the initial approach structure for the source*

$$(\mathcal{F} \rightarrow (\mathcal{F}_c, \mathcal{A}_{D_u}) : F \mapsto F \star G)_{G \in \mathcal{F}_c}.$$

Define for each cdf F and each $\alpha \in \mathbb{R}_0^+$ the mapping

$$\phi_{F, \alpha} : \mathcal{F} \rightarrow [0, 1]$$

by putting

$$\phi_{F, \alpha}(G) = \sup_{x \in \mathbb{R}} \max\{F(x - \alpha) - G(x), G(x) - F(x + \alpha)\}.$$

Furthermore, for $F \in \mathcal{F}$, let $\Phi(F)$ be the set of all maps $\phi_{F, \alpha}$, where α runs through \mathbb{R}_0^+ .

The proofs of the following results can be found in [L15].

Theorem 3.3. *The collection of sets $(\Phi(F))_{F \in \mathcal{F}}$ is a basis for the approach system of \mathcal{A}_c .*

Theorem 3.4. *The topological coreflection of \mathcal{A}_c is \mathcal{T}_w . The metric coreflection of \mathcal{A}_c is D_u .*

Theorem 3.5. *The space $(\mathcal{F}, \mathcal{A}_c)$ is locally countably generated and sequentially complete.*

It is the aim of this section to express the relative sequential compactness index of a set \mathcal{D} in the space $(\mathcal{F}, \mathcal{A}_c)$ in terms of a canonical index measuring up to what extent \mathcal{D} is tight ([K02]). We thus obtain a strong quantitative generalization of Prokhorov's Theorem (Theorem 3.13). To this end, we make use of the compactness result obtained in the previous section. First some preparation is required.

Define, for $\gamma \in \mathbb{R}_0^+$, the metric $L_\gamma(F, G)$ between F and G in \mathcal{F} as the infimum of all $\alpha \in \mathbb{R}_0^+$ for which the inequalities

$$F(x - \gamma\alpha) - \alpha \leq G(x) \leq F(x + \gamma\alpha) + \alpha$$

hold for all points $x \in \mathbb{R}$. The metric L_γ is known as the Lévy metric with parameter γ ([K02]).

Theorem 3.6. *The assignment of collections*

$$(\{L_\gamma(F, \cdot) \mid \gamma \in \mathbb{R}_0^+\})_{F \in \mathcal{F}} \quad (1)$$

is a basis for the approach system of \mathcal{A}_c .

Proof. It is easily seen that $L_{\gamma_1} \leq L_{\gamma_2}$ whenever $\gamma_2 \leq \gamma_1$, whence (1) is a basis for an approach structure which we denote \mathcal{A} . Now it is enough to prove that, for all $F \in \mathcal{F}$ and $\mathcal{D} \subset \mathcal{F}$ nonempty, $\delta_{\mathcal{A}}(F, \mathcal{D}) = \delta_{\mathcal{A}_c}(F, \mathcal{D})$. We will do this in two steps, making use of Theorem 3.3.

1) $\delta_{\mathcal{A}}(F, \mathcal{D}) \leq \delta_{\mathcal{A}_c}(F, \mathcal{D})$: If $\delta_{\mathcal{A}_c}(F, \mathcal{D}) < \theta$ with $\theta > 0$, then for $\gamma > 0$ there exists $G \in \mathcal{D}$ for which $\phi_{F, \gamma\theta}(G) < \theta$. But then we have for all real numbers x that $F(x - \gamma\theta) - G(x) < \theta$ and $G(x) - F(x + \gamma\theta) < \theta$, from which we deduce that $L_\gamma(F, G) < \theta$ and hence $\delta_{\mathcal{A}}(F, \mathcal{D}) \leq \theta$, which proves the desired inequality.

2) $\delta_{\mathcal{A}_c}(F, \mathcal{D}) \leq \delta_{\mathcal{A}}(F, \mathcal{D})$: If $\delta_{\mathcal{A}_c}(F, \mathcal{D}) > \theta$ with $\theta > 0$, then there exists $\alpha > 0$ such that for all $G \in \mathcal{D}$ we have $\phi_{F, \alpha}(G) > \theta$. If we put $\gamma = \alpha\theta^{-1}$, then it follows that for every $G \in \mathcal{D}$ there exists $x \in \mathbb{R}$ such that $F(x - \gamma\theta) - G(x) > \theta$ or $G(x) - F(x + \gamma\theta) > \theta$. We conclude that $L_\gamma(F, G) \geq \theta$ and hence $\delta_{\mathcal{A}}(F, \mathcal{D}) \geq \theta$, which proves the desired inequality. \square

. We call a finite set of points at which F is continuous an F -net and we introduce for each F -net \mathcal{N} the mapping

$$\psi_{F, \mathcal{N}} : \mathcal{F} \rightarrow [0, 1]$$

by setting

$$\psi_{F, \mathcal{N}}(G) = \sup_{x \in \mathcal{N}} |F(x) - G(x)|.$$

Lemma 3.7. *For every $F \in \mathcal{F}$ the following hold.*

1) *For an F -net \mathcal{N} and $\epsilon > 0$ there exists $\alpha \in \mathbb{R}_0^+$ so that*

$$\psi_{F, \mathcal{N}}(G) \leq \phi_{F, \alpha}(G) + \epsilon$$

for each $G \in \mathcal{F}$.

2) For $\alpha \in \mathbb{R}_0^+$ and $\epsilon > 0$ there exists an F -net \mathcal{N} so that

$$\phi_{F,\alpha}(G) \leq \psi_{F,\mathcal{N}}(G) + \epsilon$$

for each $G \in \mathcal{F}$.

Proof. Let $F \in \mathcal{F}$.

1) Fix an F -net \mathcal{N} and $\epsilon > 0$. Since all $x \in \mathcal{N}$ are continuity points of F , we may choose $\alpha \in \mathbb{R}_0^+$ such that

$$\forall x \in \mathcal{N}, \forall y \in X : |x - y| \leq \alpha \Rightarrow |F(x) - F(y)| \leq \epsilon.$$

Now, for $G \in \mathcal{F}$ and $x \in \mathcal{N}$, we have on the one hand

$$F(x) - G(x) \leq F(x - \alpha) - G(x) + \epsilon \leq \phi_{F,\alpha}(G) + \epsilon,$$

and on the other

$$G(x) - F(x) \leq G(x) - F(x + \alpha) + \epsilon \leq \phi_{F,\alpha}(G) + \epsilon,$$

from which it follows that

$$\psi_{F,\mathcal{N}}(G) \leq \phi_{F,\alpha}(G) + \epsilon$$

and we are done.

2) Fix $\alpha \in \mathbb{R}_0^+$ and $\epsilon > 0$. The number of discontinuities of F being at most countable, it is possible to construct an F -net \mathcal{N} consisting of points

$$x_0 < x_1 < \dots < x_{n-1} < x_n$$

such that $F(x_0) \leq \epsilon$, $x_{i+1} - x_i < \alpha$ for all $i \in \{0, \dots, n-1\}$ and $F(x_n) \geq 1 - \epsilon$.

Now fix $G \in \mathcal{F}$ and $x \in \mathbb{R}$. We distinguish between the following cases.

If there exists $i \in \{0, \dots, n-1\}$ such that $x_i \leq x < x_{i+1}$, then

$$F(x - \alpha) - G(x) \leq F(x_i) - G(x_i) \leq \psi_{F,\mathcal{N}}(G) + \epsilon$$

and

$$G(x) - F(x + \alpha) \leq G(x_{i+1}) - F(x_{i+1}) \leq \psi_{F,\mathcal{N}}(G) + \epsilon.$$

If $x < x_0$, then

$$F(x - \alpha) - G(x) \leq F(x_0) \leq \epsilon \leq \psi_{F,\mathcal{N}}(G) + \epsilon$$

and

$$G(x) - F(x + \alpha) \leq G(x) - F(x) \leq G(x_0) - (F(x_0) - \epsilon) \leq \psi_{F,\mathcal{N}}(G) + \epsilon.$$

If $x \geq x_n$, then

$$F(x - \alpha) - G(x) \leq F(x) - G(x) \leq (F(x_n) + \epsilon) - G(x_n) \leq \psi_{F,\mathcal{N}}(G) + \epsilon$$

and

$$G(x) - F(x + \alpha) \leq \epsilon \leq \psi_{F,\mathcal{N}}(G) + \epsilon.$$

Hence we conclude that

$$\phi_{F,\alpha}(G) \leq \psi_{F,\mathcal{N}}(G) + \epsilon,$$

which finishes the proof. \square

. For $F \in \mathcal{F}$, let $\Psi(F)$ be the set of all maps $\psi_{F,N}$, with N running through all F -nets.

Theorem 3.8. *The collection of sets $(\Psi(F))_{F \in \mathcal{F}}$ is a basis for the approach system of \mathcal{A}_c .*

Proof. Combine Theorem 3.3 and Lemma 3.7. \square

The following result provides us with information about the Lindelöf index of the space $(\mathcal{F}, \mathcal{A}_c)$.

Theorem 3.9. *We have*

$$\chi_L(\mathcal{F}, \mathcal{A}_c) = 0.$$

Proof. Fix a basis $(\mathcal{B}_F)_{F \in \mathcal{F}}$ for \mathcal{A}_c , $(\phi_F)_{F \in \mathcal{F}} \in \prod_{F \in \mathcal{F}} \mathcal{B}_F$ and $\epsilon > 0$. The space $(\mathcal{F}, \mathcal{T}_w)$ being separable ([P05]), we fix a countable set $\mathcal{D} \subset \mathcal{F}$ which is dense for the weak topology. Now, the assignment of collections

$$(\{L_{1/n}(F, \cdot) \mid n \in \mathbb{N}_0\})_{F \in \mathcal{F}}$$

being a basis for \mathcal{A}_c (Theorem 3.6), there exist for each $F \in \mathcal{F}$:

1) a number $n_F \in \mathbb{N}_0$ such that

$$\phi_F(G) < L_{1/n_F}(F, G) + \epsilon/3 \quad (2)$$

for each $G \in \mathcal{F}$

2) an element $D_F \in \mathcal{D}$ such that

$$L_{1/n_F}(F, D_F) < \epsilon/3. \quad (3)$$

Combining (2) and (3) we have, for $F, G \in \mathcal{F}$,

$$\begin{aligned} \phi_F(G) &< L_{1/n_F}(F, G) + \epsilon/3 \\ &\leq L_{1/n_F}(F, D_F) + L_{1/n_F}(D_F, G) + \epsilon/3 \\ &< L_{1/n_F}(D_F, G) + 2\epsilon/3. \end{aligned} \quad (4)$$

Now consider the function

$$\zeta : \mathcal{F} \rightarrow \mathbb{N}_0 \times \mathcal{D}$$

defined by $\zeta(F) = (n_F, D_F)$ and fix for each $(n, D) \in \zeta(\mathcal{F})$ a cdf $H_{n,D} \in \mathcal{F}$ such that $\zeta(H_{n,D}) = (n, D)$. Thus, by (4),

$$\phi_{H_{n,D}}(G) < L_{1/n}(D, G) + 2\epsilon/3 \quad (5)$$

for each $G \in \mathcal{F}$. Consider the countable set $\mathcal{C} = \{H_{n,D} \mid (n, D) \in \zeta(\mathcal{F})\}$. We claim that

$$\sup_{F \in \mathcal{F}} \inf_{C \in \mathcal{C}} \phi_C(H) \leq \epsilon.$$

Indeed, for $F \in \mathcal{F}$ it suffices to consider the point $H_{n_F, D_F} \in \mathcal{C}$ since by (2) and (5)

$$\phi_{H_{n_F, D_F}}(F) < L_{1/n_F}(D_F, F) + 2\epsilon/3 < \epsilon.$$

This finishes the proof. \square

. We call a set $\mathcal{D} \subset \mathcal{F}$ *weakly relatively sequentially compact* iff it is relatively sequentially compact under the weak topology on \mathcal{F} , i.e. each sequence in \mathcal{D} contains a weakly convergent subsequence.

. Since the weak topology is the topological coreflection of \mathcal{A}_c (Theorem 3.4), the space $(\mathcal{F}, \mathcal{A}_c)$ is locally countably generated and sequentially complete (Theorem 3.5) and $\chi_L(\mathcal{F}, \mathcal{A}_c) = 0$ (Theorem 3.9), we may apply Theorem 2.2 to conclude that

Theorem 3.10. *For a set $\mathcal{D} \subset \mathcal{F}$ we have*

$$(\chi_{rsc})_{\mathcal{A}_c}(\mathcal{D}) = (\chi_{rc})_{\mathcal{A}_c}(\mathcal{D}).$$

Furthermore,

$$\mathcal{D} \text{ is weakly relatively sequentially compact} \Leftrightarrow (\chi_{rsc})_{\mathcal{A}_c}(\mathcal{D}) = 0.$$

. Recall that a collection $\mathcal{D} \subset \mathcal{F}$ is *tight* ([K02]) iff for each $\epsilon > 0$ there exists a constant $M \in \mathbb{R}_0^+$ such that $\max\{F(-M), 1 - F(M)\} \leq \epsilon$ for all $F \in \mathcal{D}$.

. We now define the number

$$\chi_e(\mathcal{D}) = \inf_{M > 0} \sup_{F \in \mathcal{D}} \max\{F(-M), 1 - F(M)\}.$$

We call $\chi_e(\mathcal{D})$ the *escape index* of \mathcal{D} (not to be confused with the tightness indices discussed in [BLV11]). Notice that \mathcal{D} is tight if and only if $\chi_e(\mathcal{D}) = 0$.

. The following simple example shows that the escape index produces meaningful non-zero values.

Example 3.11. *Fix $0 < \alpha < 1$ and let \mathcal{F} be the set of all probability distributions $F_n = (1 - \alpha)F_{\delta_0} + \alpha F_{\delta_n}$, $n \in \mathbb{N}_0$, F_{δ_x} standing for the Dirac probability distribution making a jump of height 1 at x . Then $\chi_e(\mathcal{D}) = \alpha$.*

. We finally come to a quantitative generalization of Prokhorov's Theorem for the continuity approach structure. As in the classical case, the proof is based on Helly's Selection Principle ([K02]).

Theorem 3.12 (Helly's Selection Principle). *Fix a number $M \in \mathbb{R}_0^+$ and a sequence $(F_n : [-M, M[\rightarrow [0, 1])_n$ of non-decreasing right-continuous functions. Then there exists a subsequence $(F_{k_n})_n$ and a non-decreasing right-continuous function $F : [-M, M[\rightarrow [0, 1]$ such that $F_{k_n}(x) \rightarrow F(x)$ for each point x at which F is continuous.*

Theorem 3.13 (Quantitative Prokhorov's Theorem). *For $\mathcal{D} \subset \mathcal{F}$ we have*

$$(\chi_{rsc})_{\mathcal{A}_c}(\mathcal{D}) = \chi_e(\mathcal{D}).$$

Proof. Recall that, by Theorem 3.10,

$$(\chi_{rsc})_{\mathcal{A}_c}(\mathcal{D}) = (\chi_{rc})_{\mathcal{A}_c}(\mathcal{D}).$$

1) $(\chi_{rsc})_{\mathcal{A}_c}(\mathcal{D}) \leq \chi_e(\mathcal{D})$: Fix $\epsilon > 0$ and a sequence $(F_n)_n$ in \mathcal{D} . Now we choose a constant $M \in \mathbb{R}_0^+$ in such a way that for each $F \in \mathcal{D}$ it holds that $\max\{F(-M), 1 - F(M)\} \leq \chi_e(\mathcal{D}) + \epsilon$. Then Helly's Selection Principle furnishes a subsequence $(F_{k_n})_n$ and a non-decreasing right-continuous function $G : [-M, M[\rightarrow [0, 1]$ such that $F_{k_n}(x) \rightarrow G(x)$ for all points x at which G is continuous. Finally, we define $\tilde{G} \in \mathcal{F}$ by

$$\tilde{G}(x) = \begin{cases} 0 & \text{if } x < -M \\ G(x) & \text{if } -M \leq x < M \\ 1 & \text{if } x \geq M \end{cases}.$$

But then, by Theorem 3.8, we clearly have $\lambda_{\mathcal{A}_c}(F_{k_n} \rightarrow \tilde{G}) \leq \chi_e(\mathcal{D}) + \epsilon$ and hence $(\chi_{rsc})_{\mathcal{A}_c}(\mathcal{D}) \leq \chi_e(\mathcal{D})$.

2) $\chi_e(\mathcal{D}) \leq (\chi_{rc})_{\mathcal{A}_c}(\mathcal{D})$: Let $\epsilon > 0$. Then for $\alpha \in \mathbb{R}_0^+$ there exists a finite collection $\mathcal{E} \subset \mathcal{F}$ such that for all $F \in \mathcal{D}$ we can find $G \in \mathcal{E}$ for which $\phi_{G,\alpha}(F) \leq (\chi_{rc})_{\mathcal{A}_c}(\mathcal{D}) + \epsilon/2$. Since \mathcal{E} is finite we may choose a constant $\tilde{M} \in \mathbb{R}_0^+$ such that for each $G \in \mathcal{E}$ we have $G(-\tilde{M}) \leq \epsilon/2$ and $G(\tilde{M}) \geq 1 - \epsilon/2$. Now put $M = \tilde{M} + \alpha$, fix $F \in \mathcal{D}$ and choose $G \in \mathcal{E}$ in such a way that $\phi_{G,\alpha}(F) \leq (\chi_{rc})_{\mathcal{A}_c}(\mathcal{D}) + \epsilon/2$. Then we have on the one hand

$$\begin{aligned} F(-M) &= F(-\tilde{M} - \alpha) \\ &\leq G(-\tilde{M}) + ((\chi_{rc})_{\mathcal{A}_c}(\mathcal{D}) + \epsilon/2) \\ &\leq (\chi_{rc})_{\mathcal{A}_c}(\mathcal{D}) + \epsilon, \end{aligned}$$

and on the other

$$\begin{aligned} F(M) &= F(\tilde{M} + \alpha) \\ &\geq G(\tilde{M}) - ((\chi_{rc})_{\mathcal{A}_c}(\mathcal{D}) + \epsilon/2) \\ &\geq 1 - ((\chi_{rc})_{\mathcal{A}_c}(\mathcal{D}) + \epsilon), \end{aligned}$$

entailing that $\chi_e(\mathcal{D}) \leq (\chi_{rc})_{\mathcal{A}_c}(\mathcal{D})$. □

Corollary 3.14 (Classical Prokhorov's Theorem). *For $\mathcal{D} \subset \mathcal{F}$ the following are equivalent.*

- (1) *The collection \mathcal{D} is weakly relatively sequentially compact.*
- (2) *The collection \mathcal{D} is tight.*

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